A Selective Review on Recent Development of Displacement-Based Laminated Plate Theories

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Abstract: This paper reviews the recent development of displacement-based theories for laminated composite plates as well as corresponding finite element models. Discussion focuses on the accuracy and efficiency of various theories, and the detailed expression of typical displacement theories used herein is also presented. To objectively assess these theories, Pagano's cylindrical bending problems are chosen for comparison of various theories. Numerical results show that the global-local theories are more suitable for prediction of transverse shear stresses directly from constitutive equations in comparison with other theories. However, the zig-zag theories satisfying the interlaminar continuity of transverse shear stresses are still unable to accurately predict transverse shear stresses directly from constitutive equations, in which in order to obtain satisfactory transverse shear stresses, 3D equilibrium equations have to be adopted. In addition, free vibrations and stability of soft-core sandwiches are also considered to further assess various displacement-based laminated plate theories. Numerical results show that the global displacement theories will encounter difficulties to accurately predict the dynamic and the buckling response of so special structures. However, the global-local theories satisfying the continuity of transverse shear stresses at interfaces are still suitable for the dynamic and the buckling problems of soft-core sandwiches. In addition, this paper also includes some information of recent patents on the processes for the fabrication of composite metal object as well as functionally graded materials, and the methods of making nanofibre yarns, ribbons and sheets *et al*.

Keywords: Displacement-based theories, laminated plates, finite element models, interlaminar continuity of transverse shear stresses, constitutive equations, 3D equilibrium equations, thermal/mechanical loads, free vibration, buckling, soft-core sandwiches.

1. INTRODUCTION

Due to their high specific strength and low specific density, laminated composite and sandwich plates were widely used in the aeronautical and aerospace industries. Numerous investigators had used various models for the analysis of laminated structures. Moreover, research on laminated plate theories can be found in early review papers [1-10]. Recent review articles, such as Kant and Swaminathan [11], Carrera [12], Reddy and Arciniega [13] and Mittelstedt and Becker [14, 15], covered much of the previous research on laminated plate theories in the past decades.

Although research on laminated plate theories has been widely reviewed in previous articles above mentioned, the present article aims at the recent development of the displacement-based equivalent single layer plate theories as well as the corresponding finite element models. The contents on the layerwise theories as well as the mixed theories based on the Riessner mixed variational theorem can be excluded in present work. Readers interesting in theses theories can refer to following excellent review papers by Reddy and Robbins [10] and Carrera [16], respectively. In the year of 2007, the new patents [17-21] on composite metal object and Functionally Graded Materials *et al*. have been reported. Herein, these patents will be introduced. The patent [17] showed the methods for fabrication of the composite metal object comprised ductile crystalline metal particles in an amorphous metal matrix. The invention [18] provides processes for the fabrication of Functionally Graded Materials (FGM) by a controlled Electrophoretic Deposition (EPD). The novel non-linear vinyl polymers are composed of a multifunctional peroxide, and a cross-linking agent and/or a chain transfer agent, and methods of making such polymers have been given in the patent [19]. The patent [20] indicated that a single piezoelectric is excited at a first frequency to cause two vibration modes in a resonator producing a first elliptical motion in a first direction at a selected contacting portion of the resonator. Patent [21] presented the methods of making the nanofiber yarns, ribbons, and sheets. These patents are only introduced simply, as the focus of this review is on the recent development of laminated plate theories. Considering the scope of this review as well as the limitation of author's knowledge, authors feel regret that numerous works may have been neglected in present review article.

2. DISPLACEMENT-BASED LAMINATE THEORIES

As far as the development of displacement-based theories for laminated composite structures is concerned, four approaches are usually adopted in published literature. The first approach, which is generally called as the global displacement theories (GDT), can be obtained by expanding the displacement components as Taylor's series in terms of the thickness coordinate *z*. The second approach is the improved global displacement theory (IGDT) called here, which can be obtained by adding a zigzag-shaped C^0 function to in-plane displacement components of arbitrary global displacement theories mentioned above. It should be indicated that this theory is still unable to *a priori* satisfy continuity of transverse shear stresses at interfaces. The third approach is zig-zag theory (ZZT) which can be constructed by superimposing linear zig-zag field to the global displacement field. By imposing both continuity of transverse shear stresses at interfaces and free surface conditions, the number of unknowns in this model is independent of the number of layers. Herein, last approach is called as the global-local displacement theories (GLDT) in which one can be obtained by using double superposition hypothesis proposed by Li and Liu [22]. By employing interface continuity conditions and free surface conditions of transverse shear stresses, the number of total unknowns is independent of the layers of laminates. Characters and recent development of various theories mentioned above will be detailedly introduced in following sections.

2.1. Global Displacement Theories

Typical theories in the first approach are of the first order shear deformation theories [23-26] which assume a constant transverse shear strain across the thickness direction. Therefore, the shear correction factor is generally used to adjust the transverse shear stiffness of laminates. By using the first order theory in conjunction with a postprocessing procedure, Rolfes *et al*. [27] have predicted

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transverse thermal stresses in composite plates. In addition, Sze *et al.* [28] used finite element model based on first-order theory as well as postprocessing procedure to predict the interlaminar stresses of laminated plates. The major problem in constructing finite element based on first-order theory is how to eliminate shear locking for very thin laminated plates. Based on first-order theory, Ge and Chen [29] proposed a three-node triangular plate element for bending problems of laminated composite plates. By employing the re-constituting shear strain technique, this triangular element is free from shear locking problem. At the same year, Cen *et al*. [30] proposed a four-node quadrilateral element for analysis of laminated composite plates, which is also able to avoid shear locking problems. Recently, Liew *et al*. [31] proposed a meshfree Galerkin method based on first-order theory to analyze the buckling of folded plate structures. More comments on first-order theories can be found in following review papers [7, 10].

By gradually adding high-order terms of transverse coordinate *z* to the in-plane and transverse displacements, the higher order shear deformation theories have been widely developed. Considering transverse normal strain, Marur and Kant [32] presented a higher order theory, in which the in-plane displacement field consists of third-order polynomial in global thickness coordinate *z* whereas the transverse deflection is represented by a second-order polynomial of global coordinate *z*, to predict transient dynamic response of laminated beams. At the same year, Kant *et al.* [33] proposed a third-order theory to predict the dynamic response of laminated composite beams. Thereinto, in-plane and transverse displacement fields of the third order theory consist of third-order polynomial in global thickness coordinate *z*, respectively. Subsequently, Kant and Swaminathan [34] employed the third order theory as well as other higher-order theories [35-37] to predict that static response of laminated composite plate. In addition, by analyzing the bending behaviors of simply-supported sandwich plates with arbitrary angle-ply face sheets, Swaminathan *et al.* [38] further assessed the accuracy of several higher-order theories. Recently, Swaminathan and Patil [39] also employed several higher-order theories for the stress analysis of antisymmetric angle-ply plates. Kant and Swaminathan [40] used the higher order theories to predict the dynamic response of cross-ply laminated composite and sandwich plates. Swaminathan and Patil [41] extended the higher order theories for the free vibration analysis of antisymmetric angle-ply plates. With development of global higher-order theories, numerous finite element models have been also proposed. For example, based on a higher-order theory neglecting transverse normal strains, Kant and Khare [42] proposed a C^0 Lagrangian isoparametric faceted quadrilateral element to analyze the static behaviors of the general laminated plates and shells. Subsequently, Khare *et al.* [43] extended this higher-order facet shell element to analyze the dynamic behaviors of laminated composite plates and shells. Based on the third order theory as well as the first order theory, Babu and Kant [44] constructed two quadrilateral elements to analyze buckling behaviors of skew fibre-reinforced composite and sandwich panels. In addition, Kant and Babu [45] also extended the corresponding elements to predict thermal buckling response of skew laminated composite and sandwich plates. Based on a higher-order theory satisfying free surface conditions of transverse shear stresses [36], Reddy [46] proposed a four-noded nonconforming rectangular element for static analysis of functionally graded plates. The corresponding nonconforming element has seven degrees of freedom at per node. In addition, based on Reddy's theory, Nayak *et al*. [47] have respectively constructed both four-node and nine-node higher-order quadrilateral elements by assuming interpolations for the shear strain. It should be indicated that the two elements may overcome the shear locking. Moreover, the corresponding elements have been applied to predict the natural frequencies of isotropic, laminated composite and sandwich plates. Further, Nayak *et al.* [48] expended their finite elements to analyze the buckling and vibration of laminated composite and sandwich plates subjected to initial stresses. Based on Reddy's theory, Sheikh and Chakrabarti [49] proposed a six-node triangular plate bending elements for static analysis of laminated composite plates. It should be shown that this element is free from shear locking problem. In addition, this six-node triangular element was employed to predict buckling response of laminated composite plates [50].

At present, the higher-order theories mostly used in the literature are the third order theories. However, discussion on why the third order seems to become the upper limit of laminated theories has never been reported. To verify the effects of higher order displacement components on the dynamic and the buckling responses of thick circular rings, Matsunaga [51] proposed a ninth-order theory named as HSDT-98 herein, in which the in-plane displacement field consists of 9th-order polynomial in global thickness coordinate *z* whereas the transverse deflection is represented by an 8th-order polynomial of global coordinate *z*. Numerical results showed that higher-order shear deformations surely have an important effect on the natural frequencies and the buckling stresses of thick rings. Further, this higher-order theory (HSDT-98) as well as other higher-order theories is used to analyze the dynamic and buckling behaviors of arbitrary angle-ply laminated composite beams and plates [52-59]. In addition, Matsunaga [60-63] expended these higher-order theories to predict the static response of laminated composite and sandwich structures. Numerical results showed by increasing the order number of in-plane and transverse displacement components, the higher order theories can yield greatly improved transverse shear stresses over the available third order theories mentioned above.

One of advantages in higher-order theories is that the shear correction factor is not needed. Moreover, compared to first-order theories, higher-order theories can yield improved results over the available first order theory. However, the higher order theories are unable to account for the variation of the zig-zag form of in-plane displacement components along the thickness direction as well as violate interlaminar continuity at interfaces for the transverse stress components. Therefore, the higher order theories lose capability to accurately calculate transverse stress components directly from constitutive equations. To obtain accurate transverse stresses, the equilibrium equation method is usually adopted. Cho and Kim [64] have discussed in details the equilibrium equation approach. Moreover, they showed that the constitutive equation approach is more attractive compared to the equilibrium equation approach.

2.2. Improved Global Displacement Theories

To obtain a laminated plate theory for accounting the variation of the zig-zag form of in-plane displacement components along the thickness direction, Murakami [65] proposed to add a zigzag-shaped C^0 function to in-plane displacement components of arbitrary global displacement theories. By combining both zig-zag shape function and Legendre polynomials to approximate in-plane displacement fields, Toledano and Murakami [66] developed an improved global displacement theory. To assess performance of this theory, they analyzed the cylindrical bending problems of laminated plates. It can be found that this theory is surely able to improve the accuracy of in-plane response. Recently, Carrera [67] also discussed, with the help of numerical examples, the use of the zig-zag function proposed Murakami [65] in the global displacement theories. Moreover, a conclusion has been drawn that introduction of zig-zag function is more effective than improvement of order number in global displacement theories. Based on the global displacement theory including the zig-zag terms, Carrera [68] presented C^0 multilayered plate elements. Carrera and Demasi [69] also proposed the classical and advanced multilayered plate elements based on the global displacement theory including the zig-zag function as well as other theories. Subsequently, these elements were used to analyze the static response of multilayered plates [70]. In addition, Demasi [71] proposed a multilayered plate element based on the displacement theories by introducing Murakami's zig-zag function. More discussion on Murakami's zig-zag function can be found in following articles [72-75]. It should be indicated that the approach, in which Murakami's zig-zag function is added to global displacement theories, is unable to *a priori* satisfy interlaminar continuity for transverse stresses. Therefore, this approach can not computed transverse stresses directly from constitutive equations.

2.3. Zig-zag Theories

In the early stage of the development of these models, Di Sciuva [76] firstly proposed a linear zig-zag model which can guarantee the continuity of transverse shear stresses at interfaces. Moreover, the number of unknown variables in this model is independent of the number of layers. However, this theory is unable to satisfy the conditions of zero transverse shear stresses on the upper and lower surfaces. Subsequently, the cubic zig-zag model as well as the corresponding triangular plate element has been developed by Di Sciuva [77,78], respectively. Compared to the linear zig-zag model, the cubic zig-zag model can improve the accuracy of transverse shear stresses. However, in the cubic zig-zag model the transverse shear stresses obtained directly from the constitutive equations are still less accurate in comparison with the three-dimensional elasticity solutions. To accurately predict transverse shear stresses, Di Sciuva [78] adopted the post-processing approach of integration of equilibrium equation. Recently, Di Sciuva and Gherlone [79] developed a third-order Hermitian zig-zag model. Because the transverse shear stresses of external surfaces of the laminate are used as unknown variables, this model is suitable for analyzing the laminated beams under the distributed tangential loads on the external surfaces. Moreover, this model in conjunction with a sublaminate approach was used to study the interlayer slip problems of laminated beams [80].

Based on the zig-zag model proposed by Cho and Parmerter [81], the higher order zig-zag theories considering transverse normal strain have been developed to predict the deformation and stresses of thick smart composite plate subjected to mechanical, thermal and electric loads [82,83]. In-plane displacement components in higher order zig-zag theory consist of both a local linear zig-zag function and a global third-order displacement field. By enforcing the free conditions at upper and lower surfaces and interlaminar continuity conditions of transverse shear stresses, the number of total unknowns does not depend on the layer number of laminates. Although, this zig-zag model is able to *a priori* fulfill the continuity of transverse shear stresses at interface, the three-dimensional equilibrium equations are still needed to compute transverse shear stresses. Subsequently, a three-node triangular finite element based on the third-order zig-zag theory has been proposed by Oh and Cho [84] to predict the mechanical, thermal and electric behaviors. To obtain accurately transverse shear stresses, the stress smoothing technique within whole domain is adopted in regular meshes. However, discussion on whether the stress smoothing technique is suitable for the irregular meshes can not be reported in their study. In addition, Cho and Kim [85] have also presented a four-noded finite element in conjunction with post-process method. Based on the zig-zag theory proposed by Cho and Parmerter [86], Chakrabarti and Sheikh [87,88] developed the six-node triangular element to analyze the static behaviors of laminated and sandwich plates.

In addition, Kapuria [89] proposed a coupled zig-zag model for the static analysis of laminated composite beams with piezoelectric layers at surfaces. For this model, the mechanical component is modeled by the third-order zig-zag theory, which satisfies the free surface conditions and the geometric and stress continuity conditions at interfaces whereas the electric potential is modeled with layerwise theory. Subsequently, this model is used to predict the static behaviors of piezoelectric sandwich beams [90]. Kapuria *et al*. [91] also used the coupled zig-zag model and several global displacement models to predict the static response of the hybrid layered functionally graded beams subjected to themo-electro-mechanical loads. Recently, the coupled zig-zag model has been extended to analyze the static and the dynamic behaviors of piezoelectric laminated plates [92-94]. Kapuria *et al*. [95] used a zig-zag theory as well as global displacement theories to analyze the bending, buckling and dynamic behaviors of laminated composite and sandwich beam. Numerical results showed that the third-order theory as well as the first-order theory obviously overestimates the critical loads and natural frequencies of laminated beam with arbitrary layouts. In addition to the above zig-zag theories mentioned, the higher-order zig-zag theories considering transverse normal deformation are also developed to predict the response of laminated beams and plates under thermal loads by Kapuria *et al*. [96] and Kapuria and Achary [97], respectively. Based on the third-order zig-zag theory, Icardi [98] proposed an eight-noded plate element. To obtain accurately transverse shear stresses, 3D equilibrium equations are employed in this work. Icardi [99] also presented the C^0 plate element for global/local analysis of laminated plates based on a zig-zag model. More discussion on zig-zag theories can be found in following review articles [6, 12]. Numerical solutions show that the zig-zag models above mentioned are able to accurately calculate in-plane stresses. However, in order to accurately predict the transverse shear stress, the equilibrium equation method has to be adopted.

2.4. Global-Local Displacement Theories

From a review of the above literature, it is found that the global displacement theories, the improved global displacement theories and the zig-zag theories will encounter difficulties to accurately compute transverse shear stresses directly from constitutive equations. In view of this situation, it is desirable to present a model that can predict the transverse shear stresses directly from the constitutive equations. To this end, Li and Liu (1995) firstly attempted to propose a laminate theory based on the global-local superposition hypothesis [100]. The in-plane displacement field in this theory consists of third-order polynomial in global coordinate z along the thickness direction and order 2 power series in local coordinate ζ within each layer whereas the transverse deflection is constant along the thickness direction. However, this theory is still unable to predict transverse shear stresses directly from constitutive equations. Further, by improving the order number of local displacement in conjunction with the double-superposition hypothesis technique, Li and Liu (1997) developed the third-order global-local theory [22]. The third-order global-local theory has the potential to accurately predict the transverse shear stress components of laminated structures without any postprocessing methods. Moreover, the number of variables in this theory is independent of layer number of laminated plate. Based on this global-local theory, Sze *et al*. [101] have constructed a three-node beam element, and Chen and Chen [102] constructed triangular plate elements. After these work, the third order global-local theory can't be cared until Wu *et al* [103] and Wu and Chen [104] developed the laminated plate elements based on this model to predict thermal/mechanical response of laminated plates. Numerical examples show that third-order global-local theory is accurate for the bending problems of laminates when the number of the layers of laminates is less than six. To predict accurately the detailed response of multilayered plates, however, the fifth-order global-local theory [105] should be adopted. Moreover, the relations between the order of the global component in global-local theory and the accuracy of solutions have been studied by authors [105]. In addition, previous research [106,107] showed that the global higher-order theories obviously overestimated natural frequency for laminated composite plates with different thickness and materials at each ply and soft-core sandwich plates because these higher-order theories violate continuity conditions of the transverse stress components. In fact, our research [108] indicated that the third-order global-local theory is suitable for dynamic problems of so special structures, namely laminated composite plates with variational thickness and materials at each layer and soft-core sandwich plates. In following section, we will detailedly discuss the free vibration and the buckling problems of soft-core sandwiches.

3. DISPLACEMENT FIELDS OF SEVERAL THEORIES

To objectively assess displacement-based theories, several typical theories are chosen for comparison.

3.1. Displacement Fields of Several Typical Global Displacement Theories

3.1.1. First-Order Shear Deformation Theory (FSDT)

The first-order shear deformation theory [26], commonly known as Mindlin plate theory, predicts constant transverse shear stress through the thickness of plate. Displacement fields of first-order theory can be written as

$$\begin{aligned} u &= u_0 + z u_1 \\ v &= v_0 + z v_1 \\ w &= w_0 \end{aligned} \tag{1}$$

A shear correction factor of 5/6 is adopted in computed results using the first-order theory. Moreover, this theory only comprises 5 unknown variables.

3.1.2. Reddy's Higher-Order Shear Deformation Theory (HSDT-R)

In addition, Reddy [36] proposed a higher-order shear deformation theory which can satisfy the transverse shear stress free boundary conditions. Displacement fields of Reddy's theory herein can be written as.

$$\begin{aligned} u &= u_0 - z\frac{\partial w}{\partial x} + (z - \frac{4z^3}{3h^2})\gamma_x \\ v &= v_0 - z\frac{\partial w}{\partial y} + (z - \frac{4z^3}{3h^2})\gamma_y \\ w &= w_0 \end{aligned} \tag{2}$$

Displacement fields in Reddy's theory only consist of 5 unknown variables.

3.1.3. Higher-Order Shear Deformation Theory (HSDT-33)

To approximate the three-dimensional elasticity problem to a two-dimensional plate problem, in principle, the displacement fields can be expanded as a Taylor's series in term of the thickness coordinate. By expanding the displacement components in term of thickness coordinate, Kant and Swaminathan [34] have proposed a higher order shear deformation theory which is named as HSDT-33 herein. Both in-plane and transverse displacement fields of HSDT-33 consist of third-order polynomial in global thickness coordinate *z*, respectively.

Displacement fields of HSDT-33 can be given by

$$\begin{aligned} u &= \sum_{i=0}^{3} u_i z^i \\ v &= \sum_{i=0}^{3} v_i z^i \\ w &= \sum_{i=0}^{3} w_i z^i \end{aligned} \tag{3}$$

This model consists of 12 unknown variables.

3.1.4. Several Higher-Order Shear Deformation Theory Proposed by Matsunaga [60]

In addition to higher-order theories above mentioned, the following higher order shear deformation theories already published in previous literature are also employed for comparison. By increasing the order number of the in-plane and the transverse displacement components, Matsunaga proposed the following higher-order theories to analyze the static behaviors of laminated composite and sandwich plates. Displacement fields of HSDT-54 are

$$\begin{aligned} u &= \sum_{i=0}^{5} u_i z^i \\ v &= \sum_{i=0}^{5} v_i z^i \\ w &= \sum_{i=0}^{4} w_i z^i \end{aligned} \tag{4}$$

This model includes 17 unknown variables.

Displacement fields of HSDT-98 are

$$\begin{aligned} u &= \sum_{i=0}^{9} u_i z^i \\ v &= \sum_{i=0}^{9} v_i z^i \\ w &= \sum_{i=0}^{8} w_i z^i \end{aligned} \tag{5}$$

This model includes 29 unknown variables.

3.2. Displacement Fields of Improved Global Displacement Theory Proposed by Murakami [65]

According to the approach proposed by Murakami, the improved global theory is given by adding a Murakami's zig-zag function to global displacement fields. This theory, which is named as IGDT-M herein, can be detailedly written as follows

$$\begin{aligned} u(x,y,z) &= \sum_{i=0}^{3} z^i u_i(x,y) + M(z) u_M(x,y) \\ v(x,y,z) &= \sum_{i=0}^{3} z^i v_i(x,y) + M(z) v_M(x,y) \\ w(x,y,z) &= \sum_{i=0}^{2} z^i w_i(x,y) \end{aligned} \tag{6}$$

where *M* (*z*) denotes the Murakami's zig-zag function, which can be given by

$$M(z) = (-1)^k \zeta_k \tag{7}$$

in which, ζ_k is local thickness coordinate at *k*th ply, $-1 \le \zeta_k \le 1$.

Without any formula derivation**,** this theory can automatically satisfy the zig-zag shaped variation of in-plane displacement along thickness direction. However, this theory is unable to *a priori* fulfill the continuity conditions of transverse shear stresses at interfaces. Total number of unknown variables is only 13 in this model.

3.3. Displacement Fields of Several Refined Theories

3.3.1. Zig-Zag Theory Satisfying Continuity of Transverse Shear Stresses at Interfaces

The zig-zag theories, which satisfying continuity of transverse shear stresses at interfaces, have been proposed by superimposing a cubic varying displacement field on a zig-zag linear displacement [77,82,83]. Because these zig-zag theories mentioned can almost predict the same results, the zig-zag theory proposed by Cho and Oh [83] is only considered herein. By using transverse shear stress continuity conditions at interfaces as well as transverses shear free surface conditions, final displacement field of zig-zag theory proposed Cho and Oh [83] can be rewritten as follows

$$\begin{aligned} u^k &= u_0 + \phi_1^k(z)u_3 + \phi_2^k w_{0,x} + \phi_3^k w_{1,x} + \phi_4^k w_{2,x} \\ v^k &= v_0 + \varphi_1^k(z)v_3 + \varphi_2^k w_{0,y} + \varphi_3^k w_{1,y} + \varphi_4^k w_{2,y} \\ w^k &= w_0 + z w_1 + z^2 w_2 \end{aligned} \tag{8}$$

where, expressions of ϕ_i^k and φ_i^k can refer to reference [83]. The total number of the unknown variables is 7 in zig-zag theory. The zig-zag theory proposed by Cho and Kim is named as ZZTC in the present study.

3.3.2. Third-Order Global-Local Theory Discarding Transverse Normal Strain (TGLT-30)

The global-local higher-order theory proposed by Li and Liu [22] can *a priori* satisfy displacements and transverse shear stresses continuity conditions at interfaces. The starting displacement field can be written as follows:

$$\begin{aligned} u^k(x,y,z) &= u_G(x,y,z) + \bar{u}_L^k(x,y,z) + \hat{u}_L^k(x,y,z) \\ v^k(x,y,z) &= v_G(x,y,z) + \bar{v}_L^k(x,y,z) + \hat{v}_L^k(x,y,z) \\ w^k(x,y,z) &= w_G(x,y,z) \end{aligned} \tag{9}$$

where u_G, v_G and w_G are global components of displacement expansion; $\bar{u}_L^k$ and $\bar{v}_L^k$ are of two-term local groups; $\hat{u}_L^k$ and $\hat{v}_L^k$ are of one-term local group; the superscript *k* represents the layer order of laminated plates. The global coordinates associated with the plate are x, y, z. The reference plane ($z = 0$) is taken at the mid-plane of the laminate. The local coordinates for a layer are denoted by x, y, ζ_k where $-1 \le \zeta_k \le 1$. The relations between global coordinate and local coordinate can be seen in Fig. (**1**).

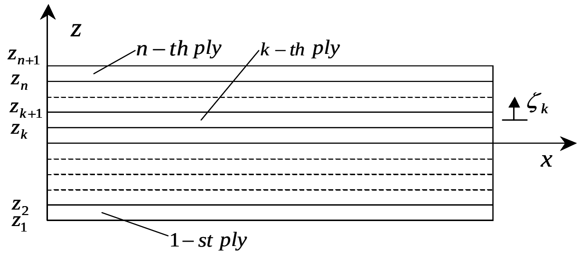

Fig. (1). Schematic figure for the laminate segment.

These global components may be written as follows

$$\begin{aligned} u_G(x,y,z) &= u_0(x,y) + z u_1(x,y) + z^2 u_2(x,y) + z^3 u_3(x,y) \\ u_G(x,y,z) &= u_0(x,y) + z u_1(x,y) + z^2 u_2(x,y) + z^3 u_3(x,y) \\ w_G(x,y,z) &= w_0(x,y) \end{aligned} \tag{10}$$

The local components can be written as

$$\begin{aligned} \bar{u}_L^k(x,y,z) &= \zeta_k u_1^k(x,y) + \zeta_k^2 u_2^k(x,y) \\ \bar{v}_L^k(x,y,z) &= \zeta_k v_1^k(x,y) + \zeta_k^2 v_2^k(x,y) \\ \hat{u}_L^k(x,y,z) &= \zeta_k^3 u_3^k(x,y) \\ \hat{v}_L^k(x,y,z) &= \zeta_k^3 v_3^k(x,y) \end{aligned} \tag{11}$$

where $\zeta_k = a_k z - b_k$; $a_k = \dfrac{2}{z_{k+1} - z_k}$; $b_k = \dfrac{z_{k+1} + z_k}{z_{k+1} - z_k}$

DISPLACEMENT CONTINUITY CONDITIONS

There are *6n+9* degrees of freedom in the above displacements. By enforcing the displacement continuity conditions proposed by Li and Liu [22], 4(*n*-1) degrees of freedom can be constrained. This especial continuity conditions can be expressed as follows

$$\begin{aligned} \bar{u}_L^k(x,y,z_k) &= \bar{u}_L^{k-1}(x,y,z_k) \\ \hat{u}_L^k(x,y,z_k) &= \hat{u}_L^{k-1}(x,y,z_k) \\ \bar{v}_L^k(x,y,z_k) &= \bar{v}_L^{k-1}(x,y,z_k) \\ \hat{v}_L^k(x,y,z_k) &= \hat{v}_L^{k-1}(x,y,z_k) \end{aligned} \tag{12}$$

where $k = 2,3,4,\cdots n$.

TRANSVERSE SHEAR STRESS CONTINUITY CONDITIONS

Using linear strain-displacement relations and three-dimensional constitutive equations for cross-ply laminated plates, the transverse shear stresses for the *kth* layer can be written as

$$\begin{aligned} \tau_{xz}^k(x,y,z) &= Q_{44k}\varepsilon_{xz}^k(x,y,z) \\ \tau_{yz}^k(x,y,z) &= Q_{55k}\varepsilon_{yz}^k(x,y,z) \end{aligned} \tag{13}$$

where Q_{ijk} are the transformed material constants for the *kth* layer; the transverse shear strains are

$$\begin{aligned} \varepsilon_{xz}^k(z) &= \frac{\partial w_0}{\partial x} + u_1 + 2zu_2 + 3z^2u_3 + a_k u_1^k + 2a_k\zeta_k u_2^k + 3a_k\zeta_k^2 u_3^k \\ \varepsilon_{yz}^k(z) &= \frac{\partial w_0}{\partial y} + v_1 + 2zv_2 + 3z^2v_3 + a_k v_1^k + 2a_k\zeta_k v_2^k + 3a_k\zeta_k^2 v_3^k \end{aligned} \tag{14}$$

By imposing the continuity conditions of transverse shear stresses at interfaces, *2(n-1)* degrees of freedom can be limited. The transverse shear stress continuity conditions are

$$\begin{aligned} \tau_{xz}^k(x,y,z_k) &= \tau_{xz}^{k-1}(x,y,z_k) \\ \tau_{yz}^k(x,y,z_k) &= \tau_{yz}^{k-1}(x,y,z_k) \end{aligned} \tag{15}$$

At present, the number of total degrees of freedom is reduced to 15. The free conditions of the transverse shear stresses on the upper and the lower surfaces are employed, finally, and the number of the independent unknowns is reduced from 15 to 11. Thereby, the final displacement models for cross-ply laminated plates are

$$u^k = u_0 + \Phi_1^k u_1^1 + \Phi_2^k u_1 + \Phi_3^k u_2 + \Phi_4^k u_3 + \Phi_5^k w_{0,x}$$
$$v^k = v_0 + \Psi_1^k v_1^1 + \Psi_2^k v_1 + \Psi_3^k v_2 + \Psi_4^k v_3 + \Psi_5^k w_{0,y} \quad (16)$$
$$w^k = w_0$$

where Φ_i^k and Ψ_i^k are the function of material constants and thickness of laminates. Expression of both Φ_i^k and Ψ_i^k can be found in reference [22]. Therefore, the total number of the unknown variables is only 11 which is independent of the number of layers in any multilayered plates.

3.3.3. Fifth-Order Global-Local Theory Discarding Transverse Normal Strain (FGLT-50)

Numerical examples showed that third-order global-local theory (TGLT) is very accurate for the bending problems of laminates as the layer number of laminates is less than six. However, with increasing of layer number, accuracy of TGLT will be gradually down. To analyze the detailed response of multilayered plates, authors have proposed a fifth-order global-local theory [105]. The fifth order global-local theory can be expressed as

$$u^k = u_0 + \Phi_1^k(z)u_1^1 + \Phi_2^k(z)u_1 + \Phi_3^k(z)u_2 + \Phi_4^k(z)u_3 + \Phi_5^k(z)u_4 + \Phi_6^k(z)u_5 + \Phi 7_5^k w_{0,x}$$
$$v^k = v_0 + \Psi_1^k(z)v_1^1 + \Psi_2^k(z)v_1 + \Psi_3^k(z)v_2 + \Psi_4^k(z)v_3 + \Psi_5^k(z)v_4 + \Psi_6^k(z)v_5 + \Psi_7^k w_{0,y}$$
$$w^k = w_0 \quad (17)$$

where, expression of both Φ_i^k and Ψ_i^k can be found in reference [105]. Thereinto, this model includes 15 unknown variables which are independent of the layer number of any multilayered plates.

3.3.4. Third-Order Global-Local Theory Considering Transverse Normal Strain (TGLT-32)

In addition, the third-order global-local theory considering transverse normal strain is also considered in this paper. This theory is suitable for predicting response of laminated plates subjected to thermal loading of uniform temperature. Displacement fields of the global-local higher-order theory, which considering transverse normal strain, are simply given by

$$u^k = u_0 + \Phi_1^k u_1^1 + \Phi_2^k u_1 + \Phi_3^k u_2 + \Phi_4^k u_3 + \Phi_5^k w_{0,x} + \Phi_6^k w_{1,x} + \Phi_7^k w_{2,x}$$
$$v^k = v_0 + \Psi_1^k v_1^1 + \Psi_2^k v_1 + \Psi_3^k v_2 + \Psi_4^k v_3 + \Psi_5^k w_{0,y} + \Psi_6^k w_{1,y} + \Psi_7^k w_{2,y}$$
$$w^k = w_0 + z w_1 + z^2 w_2 \quad (18)$$

where Φ_i^k and Ψ_i^k are the function of material constants and thickness of laminates. Expression of both Φ_i^k and Ψ_i^k can be found in reference [109]. Thus, the total number of the unknown variables is only 13 which is independent of the number of layers in any multilayered plates.

4. NUMERICAL RESULTS AND DISCUSSIONS

Among available numerical examples, Pagano's cylindrical bending problem [110] seems to be suitable and simple. Moreover, the Pagano problem is often used to assess various laminated plate theories. Thus, the laminated plates subjected to cylindrical bending are firstly considered. Subsequently, both free vibration and stability of soft-core sandwiches are also considered to assess various theories.

Example 4.1 Cylindrical bending of multilayered plate $[0^\circ/90^\circ/0^\circ/90^\circ/\cdots]$, subjected to a sinusoidal transverse loading *q=q₀sin(π x/a)*, is firstly chosen to assess several laminated plate theories see Fig. (**2**). The material properties and the thickness of each layer are uniform. The material constants, with usual notation are [110]

$$E_T = 6.89GPa,\quad E_L = 25E_T,\quad G_{LT} = 0.5E_T,\ G_{TT} = 0.2E_T,\quad \nu_{LT} = \nu_{TT} = 0.25$$

where, subscript *L* signifies the direction parallel to the fibers whereas subscript *T* denote the transverse direction. The displacements and the stresses are respectively normalized as follows

$$\bar{u} = E_2 u(0,z)/q_0 h,\ \bar{\sigma}_x = \sigma_x(a/2,z)/q_0,\ \bar{\tau}_{xz} = \tau_{xz}(0,z)/q_0$$

in which *a* is the length of laminated plates, *h* is the thickness of lamination.

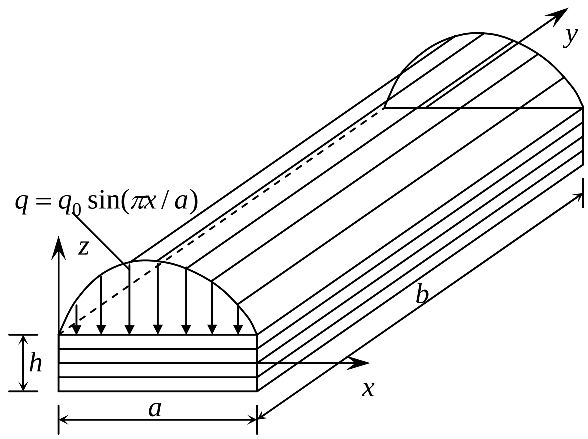

Fig. (2). Laminated plate under sinusoidal load.

To convenient comparison, acronyms have been introduced to represent the different analysis in all figures and the analytical results for all following theories have been renewedly computed by the authors. Herein, these acronyms used will be explained by

Exact: Three-dimensional elasticity solutions based on the Pagano's method [110] have been renewedly computed by the authors.

FGLT-50: Analytical results based on the fifth-order global-local theory [105].

TGLT-30: Analytical results based on the third-order global-local theory discarding transverse normal strain [22].

TGLT-32: Analytical results based on the third-order global-local theory considering transverse normal strain [109].

ZZTC: Analytical results based on the zig-zag theory proposed by Cho and Oh [83].

IGDT-M: Analytical results based on the improved global displacement theory proposed by Murakami [65].

HSDT-98: Analytical results based on the global higher order theory HSDT-98 proposed by Matsunaga [60].

HSDT-76: Analytical results based on the global higher order theory HSDT-76 proposed by Matsunaga [60].

HSDT-54: Analytical results based on the global higher order theory HSDT-54 proposed by Matsunaga [60].

HSDT-33: Analytical results based on the global higher order theory HSDT-33 proposed by Kant and Swaminathan [34].

HSDT-R: Analytical results based on the global higher order theory HSDT-R proposed by Reddy [36].

FSDT: Analytical results based on the first order theory FSDT [26].

In addition, a suffix -E, which denotes transverse shear stresses are calculated from equilibrium equations, has been also introduced to the acronyms. Other acronyms without any suffixes represent that results are computed directly from constitutive equations. Firstly, variation of in-plane displacement obtained from various theories through the thickness of four-layer plate is plotted in Figs. (**3**) and (**4**). It can be found that the displacement theories such as FGLT-50, TGLT-30 and ZZTC as well as IGDT-M proposed by Murakami [65] can describe the zig-zag shaped variation of in-plane displacement along the thickness direction. Thereinto, global-local theories such as FGLT-50 and TGLT-30 are more accurate in

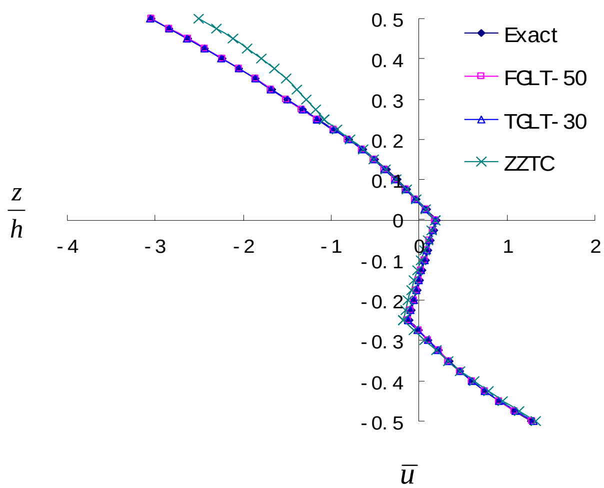

Fig. (3). Distribution of in-plane displacement obtained from refined displacement theories through thickness direction of four-layer plate (a/h = 4).

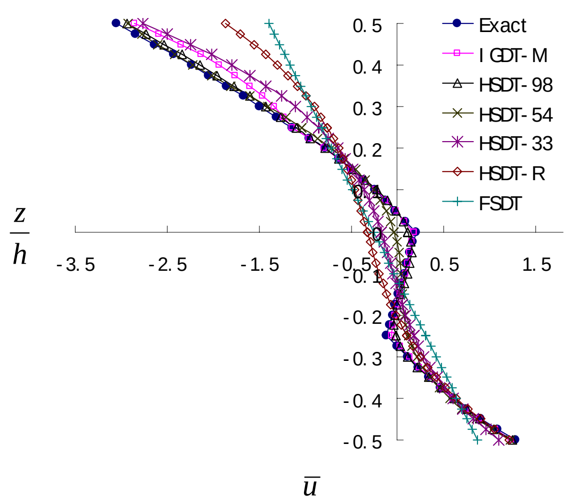

Fig. (4). Distribution of in-plane displacement obtained from several global displacement theories through thickness direction of four-layer plate (a/h = 4).

comparison with other theories mentioned. In addition, by increasing the order number of in-plane and transverse displacement components, the results obtained from the global theories (HSDT-98) are gradually close to exact solution [110]. However, the global third order theories (HSDT-33 and HSDT-R) as well as the first order theory (FSDT) are less accurate. Subsequently, distribution of in-plane stresses along the thickness direction is presented in Figs. (**5**) and (**6**). As a whole, for the prediction of in-plane stress components, the refined displacement theories (FGLT-50, TGLT-30 and ZZTC) are more accurate than other theories used herein.

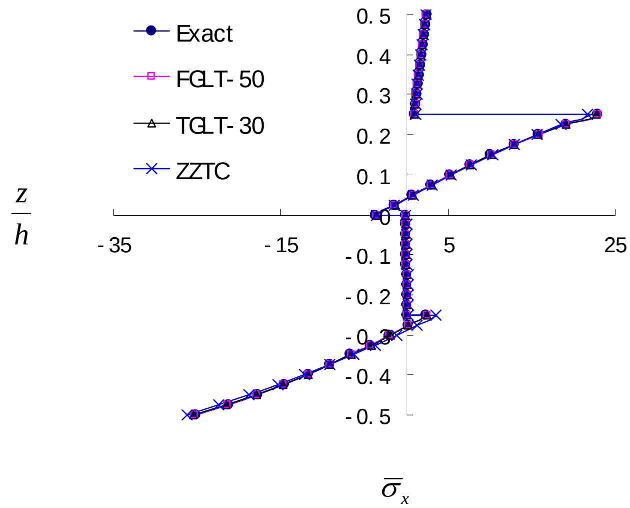

Fig. (5). Distribution of in-plane stress obtained from refined displacement theories through thickness direction of four-layer plate (a/h = 4).

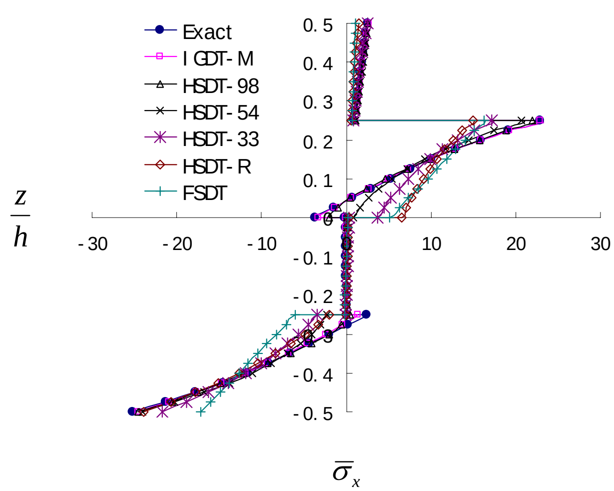

Fig. (6). Distribution of in-plane stress obtained from several displacement theories through thickness direction of four-layer plate (a/h = 4).

Transverse shear stresses obtained from the refined displacement theories are also presented in Fig. (**7**). It should be indicated that these stresses are computed directly from constitutive equations. Numerical results show that the zig-zag theory (ZZTC) proposed by Cho and Oh [83] can *a priori* satisfy the interlaminar continuity of transverse shear stresses whereas this theory is still less accurate than the global-local theories such as FGLT-50 and TGLT-30. In addition, transverse shear stresses obtained from other theories are plotted in Fig. (**8**). It is found that these transverse shear stresses calculated directly from constitutive equations are unable to satisfy the continuity at interfaces. To obtain accurately transverse shear stresses, the equilibrium equation method is adopted. Moreover, these results obtained by integrating 3D

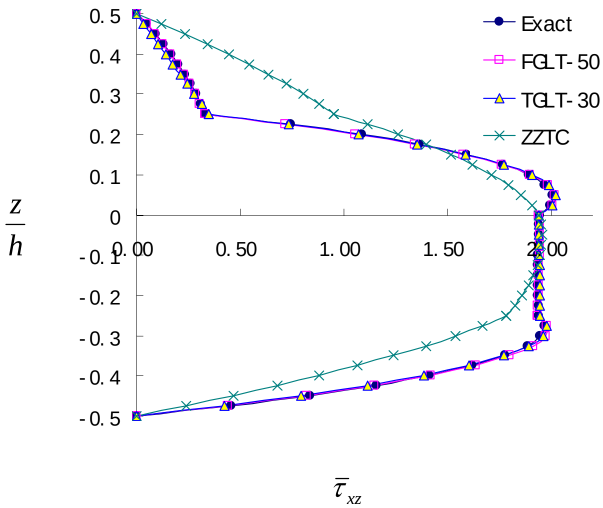

Fig. (7). Distribution of transverse shear stress obtained directly from constitutive equations through thickness direction of four-layer plate (a/h = 4).

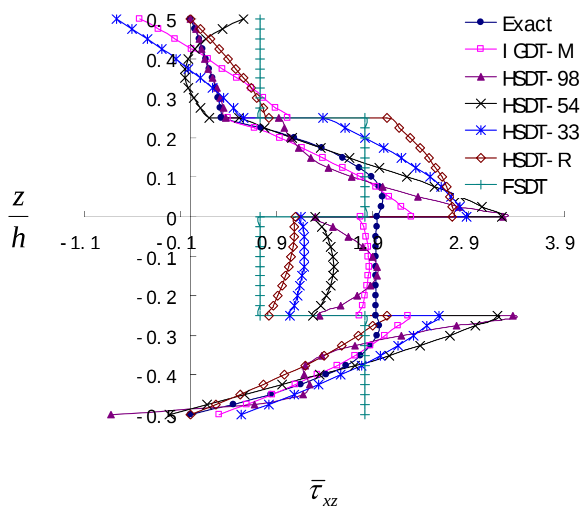

Fig. (8). Distribution of transverse shear stress obtained directly from constitutive equations through thickness direction of four-layer plate (a/h = 4).

equilibrium equation have been given in Fig. (**9**). Numerical results show that even if the equilibrium equation method is used, the third order theories (e.g., HSDT-33 and HSDT-R) as well as the first order theory are still less accurate. However, both improved global displacement theory (IGDT-M) and HSDT-98 are in good agreement with exact solution.

The static behaviors of 14-layer plate are also considered for comparison. Firstly, transverse shear stresses computed from refined displacement theories are compared with exact solution in Fig. (**10**). Numerical comparison shows that the third order global-local theory, namely TGLT-30 will encounter some difficulties for multilayered plates. However, the fifth order global-local theory (FGLT-50) is able to predict satisfactory transverse shear stresses directly from constitutive equations. In addition, the results of transverse shear stress, which computed from global displacement

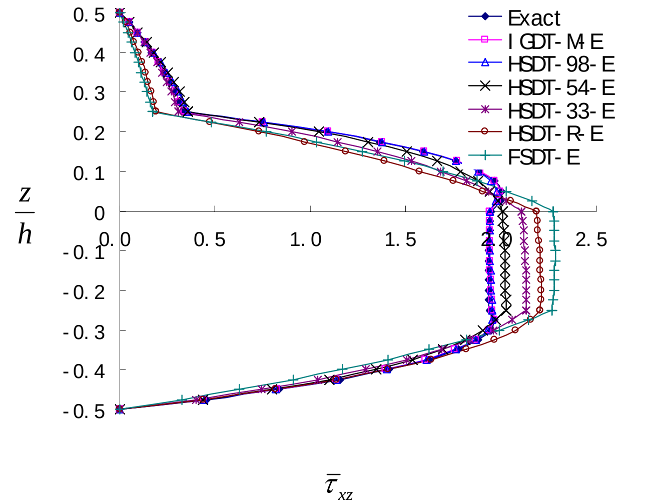

Fig. (9). Distribution of transverse shear stress obtained from equilibrium equations through thickness direction of four-layer plate (a/h = 4).

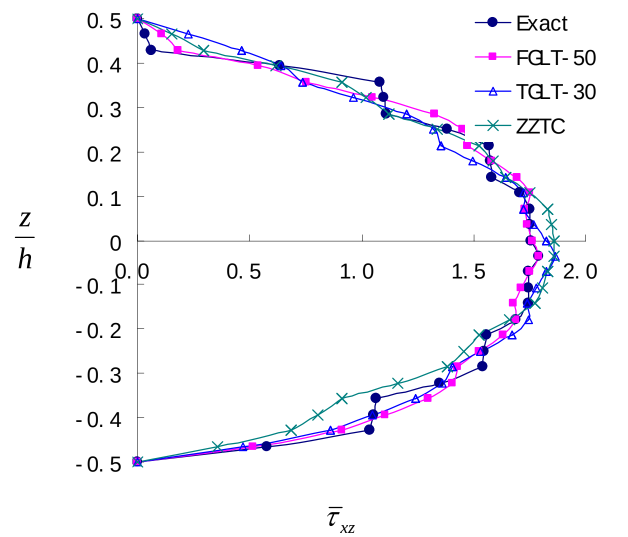

Fig. (10). Distribution of transverse shear stress obtained directly from constitutive equations through thickness direction of 14-layer plate (a/h =4).

theories by using constitutive equations, are also given in Fig. (**11**). Further, transverse shear stresses, obtained from 3D equilibrium equations, are plotted in Fig. (**12**).

Example 4.2 A simply supported laminated plate ($0^\circ/90^\circ/0^\circ$), subjected to thermal loading $T(x,y,z) = T_0 \sin(\pi x/a)\sin(\pi y/b)$ in which the temperature is uniform across the thickness direction, is analyzed, where, a and b are the length and the width of laminated plate (a= b), respectively; h is the thickness of lamination.

Material constants [63] are given as follows

$E_L/E_T = 15$; $E_T = 10GPa$; $G_{LT}/E_T = 0.5$; $G_{TT}/E_T = 0.3356$;

$\nu_{LT} = 0.3$; $\nu_{TT} = 0.49$; $\alpha_L = 0.015\times10^{-6}$; $\alpha_T = 10^{-6}$

The obtained results are nondimensionalized as follows:

$$(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xz}, \bar{\tau}_{yz}) = (\sigma_x, \sigma_y, \tau_{xz}, \tau_{yz})/(\alpha_T T_0 E_0)$$

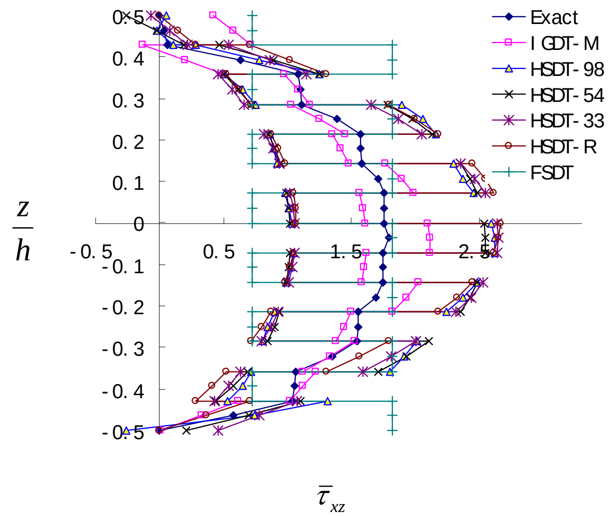

Fig. (11). Distribution of transverse shear stress obtained directly from constitutive equations through thickness direction of 14-layer plate (a/h = 4).

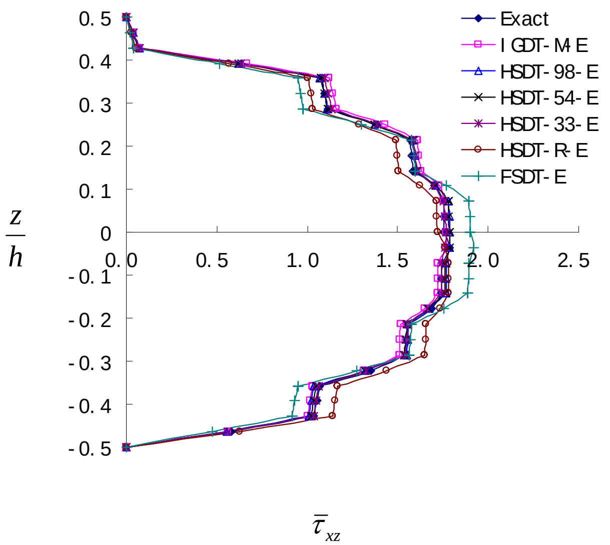

Fig. (12). Distribution of transverse shear stress obtained from equilibrium equations through thickness direction of 14-layer plate (a/h = 4).

where, $E_0 = 10$ GPa; subscript L signifies the direction parallel to the fibers whereas subscript T denote the transverse direction; α_L and α_T signify the thermal expansion coefficients along the direction parallel to the fibres and the transverse direction for laminates, respectively.

To further assess the performance of different theories used herein, a three-layer plate under uniform temperature is considered. Firstly, in-plane stresses computed by using the refined displacement theories are plotted in Fig. (**13**). Numerical results show that the third-order global-local theory considering transverse normal strain (TGDT-32) can accurately predict in-plane stresses. However, the third-order global theory discarding transverse normal strain (TGDT-30) will lose capability to accurately predict in-plane stresses. Therefore, a conclusion can be drawn that the transverse normal stain effect on the thermal expansion problems of laminated plates is very conspicuous. In addition, in-plane stresses

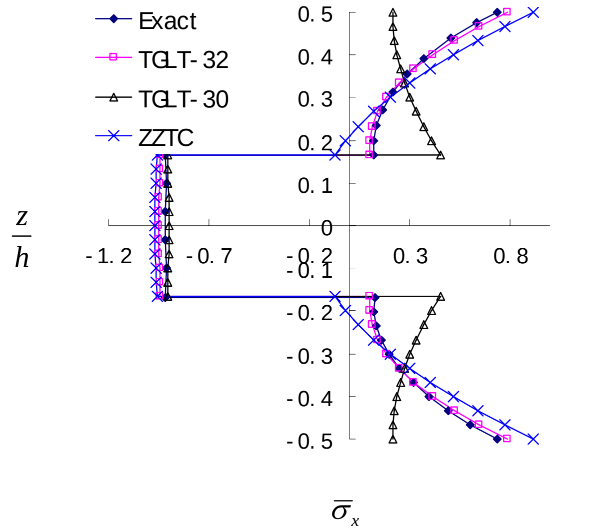

Fig. (13). Distribution of in-plane stress obtained from refined displacement theories through thickness direction of three-layer plate (a/h = 5).

obtained from several global displacement theories are compared in Fig. (**14**). Subsequently, transverse shear stresses computed using different theories directly from constitutive equations are plotted in Fig. (**15**). It can be found that transverse shear stresses, predicted from ZZTC, HSDT-R and FSDT, are zero. Even if 3-D equilibrium equations are adopted, in Fig. (**16**) it can be found that transverse shear stresses calculated using HSDT-R as well as FSDT are still unable to satisfy free surface conditions.

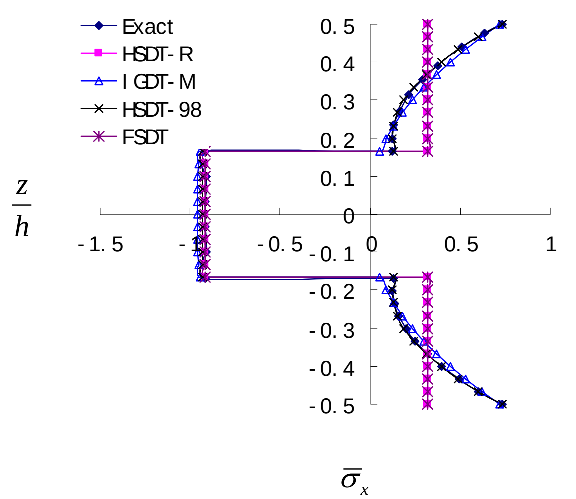

Fig. (14). Distribution of in-plane stress obtained from several displacement theories through thickness direction of three-layer plate (a/h = 5).

Example 4.3 A simply-supported plate $[0^\circ/90^\circ/0^\circ]$ under sinusoidal transverse loading $q = q_0 \sin(\pi x/a)\sin(\pi y/b)$ is analyzed.

The material properties and the thickness of each layer are uniform, and the material constants are [111]

$E_T = 6.89GPa$, $E_L = 25E_T$, $G_{LT} = 0.5E_T$, $G_{TT} = 0.2E_T$, $\nu_{LT} = \nu_{TT} = 0.25$

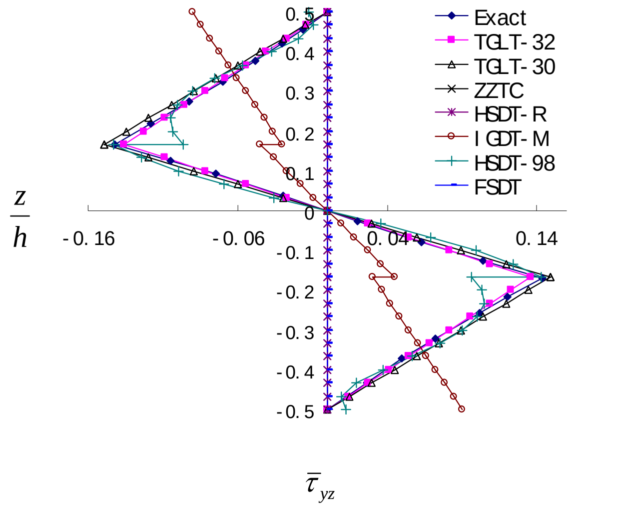

Fig. (15). Distribution of transverse shear stress obtained directly from constitutive equations through thickness direction of three-layer plate (a/h = 5).

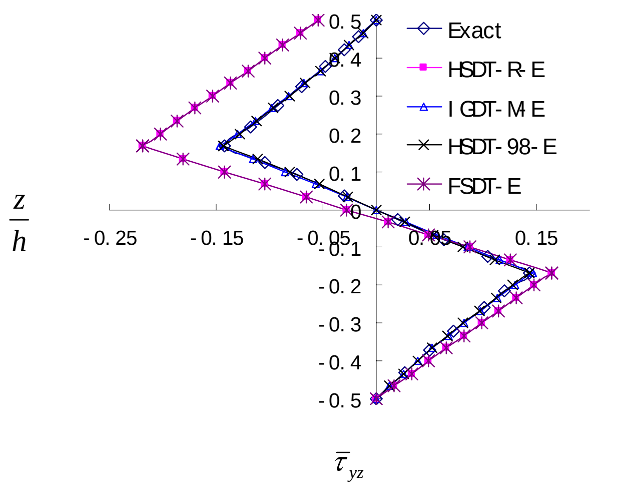

Fig. (16). Distribution of transverse shear stress obtained from equilibrium equations through thickness direction of three-layer plate (a/h = 5).

where, subscript L signifies the direction parallel to the fibers whereas subscript T denote the transverse direction. The displacements and the stresses are respectively normalized as follows

$$\bar{\sigma}_x = \sigma_x,(0,0,z)h^2 / q_0 a^2,\ \bar{\tau}_{xz} = \tau_{xz}(a/2,0,z)h / q_0 a$$

in which a and b are the length and width of laminated plates, respectively; h is the thickness of lamination.

Several displacement-based finite element models are chosen for comparison. They are

Wu & Chen: The finite element results based on the global-local higher order theory [22] are given by Wu and Chen [103] from constitutive equations. The three-noded triangular plate element has thirteen degrees of freedom in each node. For this example, the 8×8 mesh density is used.

ZZTC-Di Sciuva: The finite element results are given by Di Sciuva [78] from constitutive equations. The three-noded triangular bending element has ten degrees of freedom in each node. The 5×5 mesh density is used.

Cho & Kim: The finite element results are given by Cho and Kim [85] from constitutive equations. The 8×8 mesh density is used.

ZZTC-C&S: The finite element results based on the zig-zag theory [86] are given by Chakrabarti and Sheikh [87] from constitutive equations. The six-noded triangular bending element has seven degrees of freedom in each node. The 8×8 mesh density is used.

Carrera: The finite element results based on the global displacement theory including the zig-zag function are given by Carrera [68]. The nine-noded quadrilateral plate element has seven degrees of freedom in each node. The 2×2 mesh density is used.

Reddy-C&S: The finite element results based on Reddy's theory [36] are given by Chakrabarti and Sheikh [87] from constitutive equations. The six-noded triangular bending element has seven degrees of freedom in each node. The 8×8 mesh density is used.

FSDT-C&S: The finite element results based on the first-order theory are given by Chakrabarti and Sheikh [87] from constitutive equations. The 8×8 mesh density is used.

Exact: Three-dimensional elasticity solutions presented by Pagano [111].

This plate problem is often chosen to assess the performance of the displacement-based finite element models. Firstly, in-plane stresses computed from different finite element models are compared in Fig. (**17**). Subsequently, Fig. (**18**) shows the comparison between transverse shear stresses obtained from the finite element models and the three-dimensional elasticity solution [111]. Numerical results show that transverse shear stresses Wu&Chen in Fig. (**18**), which are calculated from the triangular plate element proposed by Wu and Chen [103] based on the global-local higher-order theory, are more accurate than other finite element results. This example shows that the accuracy of displa-cement-based finite element models depends on the performance of the displacement-based laminated plate theories.

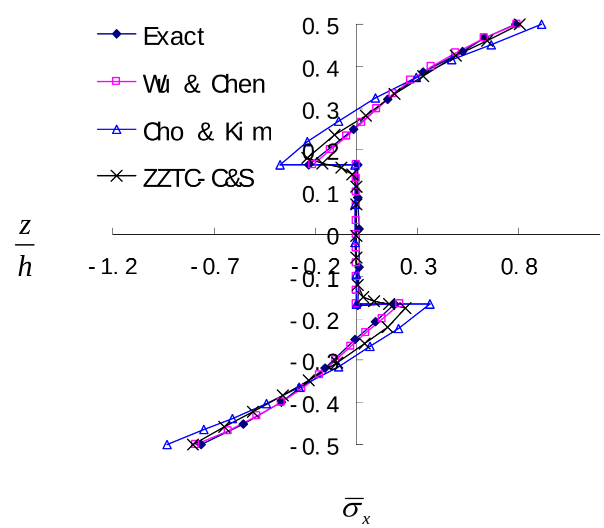

Fig. (17). Distribution of in-plane stress obtained from several displacement theories through thickness direction of three-layer plate (a/h = 4).

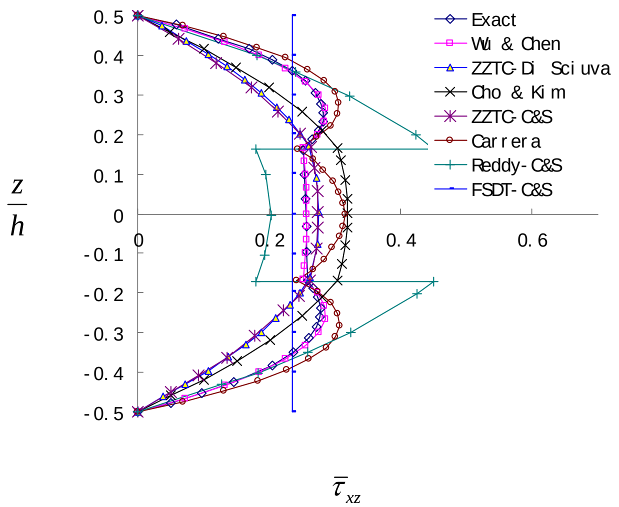

Fig. (18). Distribution of transverse shear stress obtained from constitutive equations through thickness direction of three-layer plate (a/h = 4).

Example 4.4 Free vibration analysis of a five-layer $(0^\circ/90^\circ/\text{core}/0^\circ/90^\circ)$ simply- supported soft-core sandwich plate.

The following material properties in the direction of principal material axes [107] are adopted:

Face sheets: $E_1 = 131GPa$, $E_2 = E_3 = 10.34GPa$, $G_{12} = G_{23} = 6.895GPa$,

$G_{13} = 6.205GPa$, $\nu_{12} = \nu_{13} = 0.22$, $\nu_{23} = 0.49$, $\rho = 1627kg/m^3$;

Core (isotropic): $E_1 = E_2 = E_3 = 6.89 \times 10^{-3}GPa$, $G_{12} = G_{13} = G_{23} = 3.45 \times 10^{-3}GPa$

$\nu_{12} = \nu_{13} = \nu_{23} = 0$, $\rho = 97kg/m^3$.

where the subscripts 1, 2 and 3 represent the direction of principal material axes, respectively.

In this case, a and b are the length and width of laminated plate, respectively ($a = b$); h is the thickness of laminates; t_c and t_f are the thickness of core and face sheet, respectively. The natural frequencies are normalized as $\Omega = \omega a^2 (\rho/E_2)_f^{1/2}/h$.

Analytical results from other researchers have been selected for comparison. They are

Exact [107]: Exact analytical solutions based on the propagator matrix method are given by Rao *et al*. [107].

Reddy [40]: Based on Reddy's third order theory [36], analytical solutions are given by Kant and Swaminathan [40].

FSDT [40]: Based on the first order theory [26], analytical solutions are given by Kant and Swaminathan [40].

K&S [40]: Based on the third order theory considering transverse normal strain, analytical solutions are given by Kant and Swaminathan [40].

Firstly, Table **1** presents comparison of fundamental frequencies of simply-supported soft-core sandwich plates for varying length-to-thickness ratios (a/h). In addition, the % errors of various theories relative to exact solutions [107] are also given in bracket. It can be found that results TGLT-30 agree well with exact solutions [107]. However, other global higher-order theories as well as the first order theory overestimate the natural frequencies. It should be indicated that numerical integration along thickness direction is adopted to compute the stiffness matrix and the mass matrix in present work. Therefore, a bit of difference between the results (Reddy [40] and FSDT [40]) presented by Kant and Swaminathan and the present results (HSDT-R and FSDT) can be seen in Table **1**. Due to unobvious difference, this example can still prove that present analytical method is valid. In addition, Fig. (**19**) presents the clear comparison of the % error of various theories relative to exact solutions.

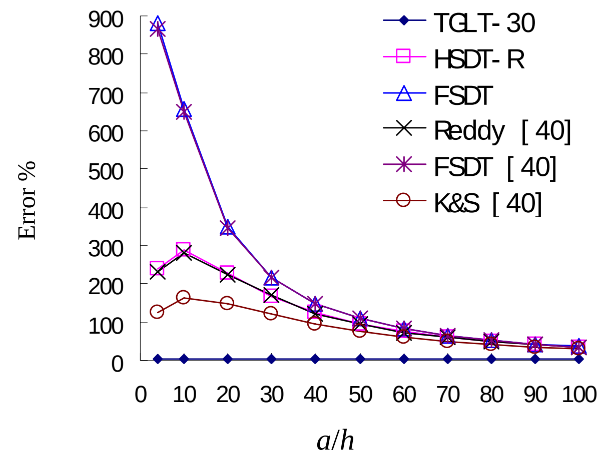

Fig. (19). Comparison of fundamental frequencies obtained from various theories for varying length-to-thickness ratios (a/h), $(0^\circ/90^\circ/\text{core}/0^\circ/90^\circ$, t_c/t_f = 10).

On the other hand, Table **2** presents comparison of the fundamental frequencies for varying thickness of the core to thickness of the face sheet (t_c/t_f). Here again, the results TGLT-30 are very close to exact solutions [107]. Moreover, the % error of various theories relative to exact solutions is plotted in Fig. (**20**).

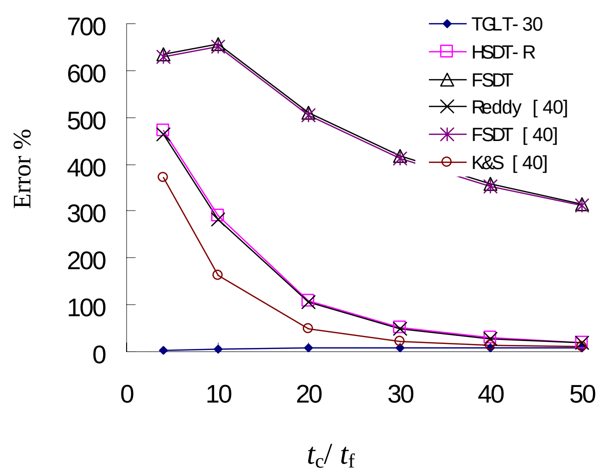

Fig. (20). Comparison of fundamental frequencies obtained from various theories, $(0^\circ/90^\circ/\text{core}/0^\circ/90^\circ$, a/h = 10).

Example 4.5 Buckling analysis of simply supported soft-core sandwich beams subjected to axial compression.

The following material properties [112] are use:

Isotropic aluminum face sheets: $E = 70GPa$, $\nu = 0.3$;

Table 1. Comparison of Fundamental Frequencies of Simply-Supported Soft-Core Sandwich Plates with Cross-Ply Face Sheets ($0^\circ/90^\circ$/ core $/0^\circ/90^\circ$, t_c/t_f =10)

a/h	Exact [107]	TGLT-30	HSDT-R	FSDT	Reddy [40]	FSDT [40]	K&S [40]
4	0.9363	0.9700 (3.5993)	3.1778 (239.3998)	9.2026 (882.8687)	3.1013 (231.2293)	9.0312 (864.5626)	2.1036 (124.6716)
10	1.8480	1.9420 (5.0866)	7.1959 (289.3885)	13.9972 (657.4242)	7.0473 (281.3474)	13.8694 (650.5087)	4.8594 (162.9545)
20	3.4791	3.6606 (5.2169)	11.4180 (228.1883)	15.6106 (348.6965)	11.2664 (223.8309)	15.5295 (346.3654)	8.5955 (147.0610)
30	5.0371	5.2882 (4.9850)	13.4800 (167.6143)	15.9825 (217.2957)	13.6640 (171.2672)	15.9155 (215.9655)	11.0981 (120.3272)
40	6.4634	6.7632 (4.6384)	14.5204 (124.6558)	16.1198 (149.4012)	14.4390 (123.3964)	16.0577 (148.4404)	12.6821 (96.2141)
50	7.7355	8.0643 (4.2505)	15.0920 (95.1005)	16.1847 (109.2263)	15.0323 (94.3287)	16.1264 (108.4726)	13.6899 (76.9750)
60	8.8492	9.1903 (3.8546)	15.4325 (74.3943)	16.2203 (83.2968)	15.3868 (73.8779)	16.1612 (82.6289)	14.3497 (62.1582)
70	9.8118	10.1524 (3.4713)	15.6495 (59.4967)	16.2419 (65.5344)	15.6134 (59.1288)	16.1845 (64.9493)	14.7977 (50.8153)
80	10.6368	10.9680 (3.1137)	15.7953 (48.4967)	16.2560 (52.8279)	15.7660 (48.2213)	16.1991 (52.2930)	15.1119 (42.0719)
90	11.3408	11.6569 (2.7873)	15.8978 (40.1824)	16.2657 (43.4264)	15.8724 (39.9584)	16.2077 (42.9150)	15.3380 (35.2462)
100	11.9400	12.2382 (2.4975)	15.9723 (33.7714)	16.2726 (36.2864)	15.9522 (33.6030)	16.2175 (35.8250)	15.5093 (29.8936)

The number in bracket () is the % errors of various theories relative to exact solutions.

Table 2. Comparison of Fundamental Frequencies of Simply-Supported Soft-Core Sandwich Plates with Cross-Ply Face Sheets ($0^\circ/90^\circ$/ Core $/0^\circ/90^\circ$, *a/h* =10)

t_c/t_f	Exact [107]	TGLT-30	HSDT-R	FSDT	Reddy [40]	FSDT [40]	K&S [40]
4	1.9084	1.9466 (2.0017)	10.8959 (470.9442)	14.0363 (635.5009)	10.7409 (462.8223)	13.9190 (629.3544)	8.9948 (371.3268)
10	1.8480	1.9420 (5.0866)	7.1959 (289.3885)	13.9972 (657.4242)	7.0473 (281.3474)	13.8694 (650.5087)	4.8594 (162.9545)
20	2.1307	2.2831 (7.1526)	4.4585 (109.2505)	13.0193 (511.0339v	4.3734 (105.2565)	12.8946 (505.1814)	3.1435 (47.5337)
30	2.3321	2.5146 (7.8256)	3.5295 (51.3443)	12.0937 (418.5755)	3.4815 (49.2861)	11.9760 (413.5286)	2.8481 (22.1260)
40	2.4690	2.6694 (8.1166)	3.1943 (29.3763)	11.3150 (358.2827)	3.1664 (28.2463)	11.2036 (353.7708)	2.8266 (14.4836)
50	2.5658	2.7777 (8.2586)	3.0735 (19.7872)	10.6615 (315.5234)	3.0561 (19.1090)	10.5557 (311.4000)	2.8625 (11.5636)

The number in bracket () is the % errors of various theories relative to exact solutions.

Orthotropic core: $E_1 = 1\times10^{-8}GPa$, $E_2 = 1.09GPa$, $G_{12} = 0.266GPa$, $v_{12} = 1\times10^{-5}$.

The critical buckling stresses are normalized as $\Lambda = \lambda L^2 / E_{2f}h^3$, where L is the length of sandwich beam; h is the thickness of laminates; t_c and t_f are the thickness of core and face sheet, respectively. Moreover, an analytical solution obtained from the mixed layerwise theory is chosen for comparison, namely

D&D(LW) [112]: An analytical results obtained from the mixed layerwise theories for stability of laminated beams is given by Dafedar and Desai [112].

The results for various values of t_c/t_f and L/h are compared with analytical results obtained from the mixed layerwise theories [112] in Table **3**. It is well known that mixed layerwise theory (LW) [112] is accurate enough to predict buckling response, so the percentage errors of various theories relative to the results of mixed layerwise theory have been given in bracket of Table **3**. To clearly compare the % error of various theories, the part results in Table **3** are plotted in Figs. (**21** and **22**). Numerical results show that the maximum percentage errors in GLHT are less than 10 whereas the maximum % errors in HSDT-33 as well as HSDT-R are more than 1000. In addition, the maximum % error of FSDT is even close to 4000. Therefore, we can draw a conclusion that the FSDT as well as the third-order theories very much overestimate the critical loads of the soft-core sandwich beams. By increasing the order number of in-plane and transverse displacement components, the maximum % errors of global higher order theories are gradually reduced. For example, the maximum % error of HSDT-54 is 101.1 whereas the maximum % error of HSDT-98 is only 34.72. Therefore, if global higher order theory is considered to predict the critical loads of soft-core sandwich beams, HSDT-98 should be at least employed.

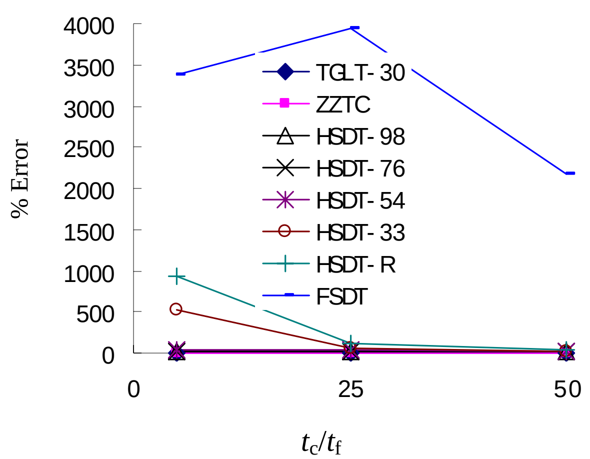

Fig. (21). Comparison of critical loads of various theories relative to analytical results obtained from mixed layerwise theory for soft-core sandwich beams (L/h = 2).

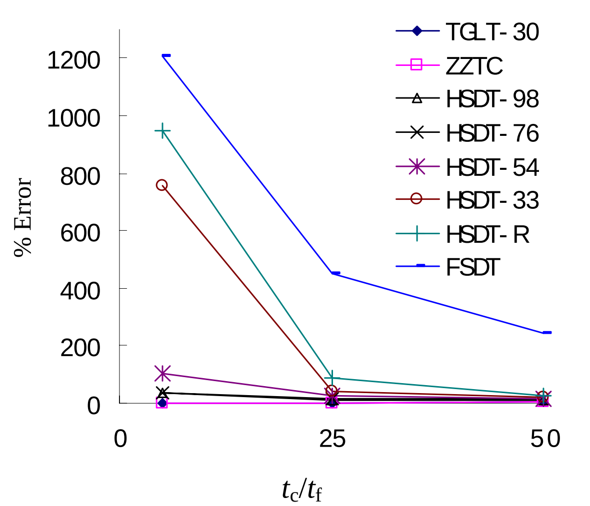

Fig. (22). Comparison of critical loads of various theories relative to analytical results obtained from mixed layerwise theory for soft-core sandwich beams (L/h = 10).

5. CURRENT & FUTURE DEVELOPMENTS

To efficiently assess displacement-based theories, this paper firstly reviews the recent development of both the displacement theories and the corresponding finite element models. Subsequently, several numerical examples are chosen to verify the accuracy and efficiency of displacement theories discussed. Based on current literature survey and numerical comparison, the following conclusions have been drawn by

1. As the layer number of laminates is less than six, the third-order global-local theory (TGLT-30) proposed by Li and Liu has capability to predict satisfactory transverse shear stresses directly from constitutive equations without any postprocessing methods. However, for bending problems of multi-layered plates, the fifth-order global-local theory (FGLT-50) should be used. In addition, the global-local theory is able to predict accurately the dynamic and the buckling response of soft-core sandwich structures. However, global higher-order theories as well as first-order theory obviously overestimate the natural frequencies and the critical loads of so special structures.
2. For the thick laminated plates under uniform temperature, the global-local theory (TGLT-30) proposed by Li and Liu discarding transverse normal deformation will lose the capability to accurately predict in-plane and transverse shear stresses. Due to inaccurate in-plane stresses, it is difficult to obtain the transverse shear stresses by integration of the 3D equilibrium equation. To accurately predict the thermal response of laminated plates under uniform temperature, authors propose to use TGLT-32.
3. The zig-zag theories (ZZTC) satisfying interlaminar continuity of transverse shear stresses at interfaces is still unable to accurately compute transverse shear stresses directly from constitutive equations. To obtain accurately transverse shear stresses, 3D equilibrium equations have to be adopted.
4. Without any formula derivation**,** the improved global displacement theory (IGDT-M) proposed by Murakami can automatically satisfy the zig-zag shaped variation of in-plane displacement through thickness direction. However, this theory is unable to satisfy interlaminar continuity of transverse shear stresses at interfaces.
5. Even if 3D equilibrium equations were adopted, Reddy's theory (HSDT-R) as well as first-order theory (FSDT) is still unable to accurately compute transverse shear stresses of both moderately thick and thick laminated plates. To obtain satisfactory transverse shear stresses, the global displacement theories with very higher order shear deformation (HSDT-98, HSDT-76 and HSDT-54) should be adopted.
6. For all two-dimensional displacement-based theories mentioned herein, only both the global-local higher-order theories and the zig-zag theories can *a priori* satisfy transverse shear stress continuity at interfaces, but neglecting the continuity condition of transverse normal stresses at interfaces. In fact, the displacement-based theories satisfying simultaneously the continuity of the transverse shear stresses as well as trans-

Table 3. Comparison of Critical Loads as Well as % Error of Various Theories Relative to Analytical Results Computed from Mixed Layerwise Theory for Soft-Core Sandwich Beam

t_c/t_f	L/h	D&D (LW) [112]	TGLT-30	HSDT-98	HSDT-76	HSDT-54	HSDT-33	HSDT-R	FSDT
5	2	0.006222	0.006719 (7.988)	0.007216 (15.98)	0.007358 (18.26)	0.008945 (43.76)	0.03851 (518.9)	0.06434 (934.1)	0.2169 (3386)
	5	0.01432	0.01484 (3.631)	0.01864 (30.17)	0.01875 (30.94)	0.02692 (87.99)	0.1643 (1047)	0.2486 (1636)	0.4499 (3042)
	10	0.041084	0.04182 (1.792)	0.05535 (34.72)	0.05571 (35.60)	0.08263 (101.1)	0.3507 (753.62)	0.4313 (949.8)	0.5367 (1206)
	50	0.34319	0.3648 (6.297)	0.4076 (18.77)	0.4086 (19.06)	0.4588 (33.69)	0.5601 (63.20)	0.5668 (65.16)	0.5728 (66.91)
25	2	0.0015299	0.001601 (4.647)	0.001777 (16.15)	0.001810 (18.31)	0.002058 (34.52)	0.002320 (51.64)	0.003343 (118.5)	0.06187 (3944)
	5	0.0090314	0.009142 (1.225)	0.01017 (12.61)	0.01037 (14.82)	0.01188 (31.54)	0.01322 (46.38)	0.01889 (109.2)	0.1395 (1444)
	10	0.031096	0.03168 (1.878)	0.03477 (11.82)	0.03538 (13.78)	0.03977 (27.89)	0.04343 (39.66)	0.05785 (86.04)	0.1717 (452.2)
	50	0.14385	0.1558 (8.307)	0.1586 (10.25)	0.1591 (10.60)	0.1624 (12.89)	0.1647 (14.49)	0.1711 (18.94)	0.1857 (29.09)
50	2	0.0014419	0.001510 (4.723)	0.001628 (12.91)	0.001709 (18.52)	0.001803 (25.04)	0.001820 (26.22)	0.001958 (35.79)	0.03281 (2175)
	5	0.0085553	0.008692 (1.5978)	0.009333 (9.0903)	0.009777 (14.28)	0.01029 (20.276)	0.01041 (21.679)	0.01113 (30.095)	0.07471 (773.26)
	10	0.026762	0.02756 (2.982)	0.02916 (8.961)	0.03024 (12.99)	0.03146 (17.56)	0.03174 (18.60)	0.03339 (24.77)	0.09229 (244.8)
	50	0.083230	0.09072 (8.999)	0.09142 (9.840)	0.09186 (10.37)	0.09229 (10.89)	0.09249 (11.13)	0.09285 (11.56)	0.09997 (20.11)

The number in bracket () is the % errors of various theories relative to analytical results D&D (LW) obtained from the mixed layerwise theory.

verse normal stress at interfaces seem to be scarce, which will be explored in our future work. Further, these theories will be used for materially and geometrically nonlinear analysis of laminated anisotropic plates.

REFERENCES

[1] Altenbach H. Theories for laminated and sandwich plates. A review. Mech Compos Mater 1998; 34: 243-252.

[2] Grigorenko YM, Vasilenko AT. Solution of problems and analysis of the stress strain state of non-uniform anisotropic shells. Int Appl Mech 1997; 33: 851-880.

[3] Jemlelita G. On kinematical assumptions of refined theories of plates: A survey. J Appl Mech 1990; 57: 1080-1091.

[4] Kapania RK, Raciti S. Recent advances in analysis of laminated beams and plates. AIAA Stud J 1989; 27(7): 923-946.

[5] Leissa AW. A review of laminated composite plate buckling. Appl Mech Rev 1987; 40(5): 575-591.

[6] Liu DS, Li XY. An overall view of laminate theories based on displacement hypothesis. J Compos Mater 1996; 30: 1539-1561.

[7] Noor AK, Burton WS. Stress and free vibration analysis of multilayered composite plates. Compos Struct 1989; 11: 183-204.

[8] Noor AK, Burton WS, Bert CW. Computational models for sandwich panels and shells. Appl Mech Rev 1996; 49(3): 155-199.

[9] Reddy JN. Mechanics of laminated composite plates theory and analysis: CRC press. 1997.

[10] Reddy JN, Robbins DH. Theories and computational models for composite laminates. Appl Mech Rev 1994; 47(6): 147-165.

[11] Kant T, Swaminathan K. Estimation of transverse/interlaminar stresses in laminated composites- a selective review and survey of current developments. Compos Struct 2000; 49: 65-75.

[12] Carrera E. Historical review of Zig-Zag theories for multilayered plates and shells. Appl Mech Rev 2003; 56(3): 287-308.

[13] Reddy JN, Arciniega RA. Shear deformation plate and shell theories: From Stavsky to Present. Mech Advanced Mater Struct 2004; 11: 535-582.

[14] Mittelstedt C, Becker W. Interlaminar stress concentrations in layered structures: Part I-A selective literature survey on the free-edge effect since 1967. J Compos Mater 2004; 38(12): 1037-1062.

[15] Mittelstedt C, Becker W. Interlaminar stress concentrations in layered structures: Part II-closed-form analysis of stresses at laminated rectangular wedges with arbitrary non-orthotropic layup. J Compos Mater 2004; 38(12): 1063-1090.

[16] Carrera E. Developments, ideas, and evaluations based upon Reissner's Mixed Variational Theorem in the modeling of multilayered plates and shells. Appl Mech Rev 2001; 54(4): 301-329.

[17] Kim, C. P., Hays, C. C., Johnson, W. L.: US200727244321 (**2007**).

[18] Olevsky, E., Wang, X., Stern, M.: WO07092363 (**2007**).

[19] Stiles, B., Sosa, J. M., Chevillard, C., Griffith, A., Remiers, J.: WO07024637 (**2007**).

[20] Magnussen, B., Varadi, P., Hagemann, B., Schofield, S., Davidson, E. M.: US200727173362 (**2007**).

[21] Zhang, M., Fang, S., Baughman, R.H., Zakhidov, A. A., Atkinson, K. R., Aliev, A. E., Li, S., Williams, C.: WO07015710 (**2007**).

[22] Li XY, Liu D. Generalized laminate theories based on double superposition hypothesis. Int J Numer Methods Eng 1997; 40: 1197-1212.

[23] Reddy JN. On refined computational models of composite laminates. Int J Numer Methods Eng 1989; 27: 361-382.

[24] Whitney JM. The effect of transverse shear deformation on the bending of laminated plates. J Compos Mater 1969; 3: 534-547.

[25] Whitney JM. Shear correction factors for orthotropic laminates under static load. J Appl Mech 1973; 40: 302-304.

[26] Whitney JM, Pagano NJ. Shear deformation in heterogeneous anisotropic plates. J Appl Mech 1970; 37: 1031-1036.

[27] Rolfes R, Noor AK, Sparr H. Evaluation of transverse thermal stresses in composite plates based on first-order shear deformation theory. Comput Methods Appl Mech Eng 1998; 167: 355-368.

[28] Sze KY, He L W, Cheung YK. Predictor-corrector procedures for analysis of laminated plates using standard Mindlin finite element methods. Compos Struct 2000; 51: 171-192.

[29] Ge ZJ, Chen WJ. A refined discrete triangular Mindlin element for laminated composite plates. Struct Eng Mech 2002; 14(5): 575-593.

[30] Cen S, Long YQ, Yao Z. H. A new hybrid-enhanced displacement-based element for the analysis of laminated composite plates. Comput Struct 2002; 80: 819-833.

[31] Liew KM, Peng LX, Kitipornchai S. Buckling of folded plate structures subjected to partial in-plane edge loads by the FSDT meshfree Galerkin method. Int J Numer Methods Eng 2006; 65: 1495-1526.

[32] Marur SR, Kant T. Transient dynamics of laminated beams: an evaluation with a higher-order refined theory. Compos Struct 1998; 41: 1-11.

[33] Kant T, Marur SR, Rao GS. Analytical solution to the dynamic analysis of laminated beams using higher order refined theory. Compos Struct 1998; 40: 1-9.

[34] Kant T, Swaminathan K. Analytical solutions for the static analysis of laminated composite and sandwich plates based on a higher order refined theory. Compos Struct 2002; 56: 329-344.

[35] Pandya BN, Kant T. Finite element stress analysis of laminated composite plates using higher order displacement model. Compos Sci Technol 1988; 32: 137-155.

[36] Reddy JN. A simple higher-order theory for laminated composite plates. J Appl Mech 1984; 51(12): 745-752.

[37] Senthilnathan NR, Lim KH, Lee KH, Chow ST. Buckling of shear deformation plates. AIAA Stud J 1987; 25: 1268-1271.

[38] Swaminathan K, Patil SS, Nataraja MS, Mahabaleswara KS. Bending of sandwich plates with anti-symmetric angle-ply face sheets-analytical evaluation of higher order refined computational models. Compos Struct 2006; 75: 114-120.

[39] Swaminathan K, Patil SS. Higher order refined computational model with 12 degrees of freedom for the stress analysis of antisymmetric angle-ply plates-analytical solutions. Compos Struct 2007; 80: 595-608.

[40] Kant T, Swaminathan K. Analytical solutions for free vibration of laminated composite and sandwich plates based on a higher-order theory. Compos Struct 2001; 53: 73-85.

[41] Swaminathan K, Patil SS. Analytical solutions using a higher order refined computational model with 12 degrees of freedom for the free vibration analysis of antisymmetric angle-ply plates. Compos Struct 2007; on line.

[42] Kant T, Khare R. A higher-order facet quadrilateral composite shell element. Int J Numer Methods Eng 1997; 40: 4477-4499.

[43] Khare R, Kant T, Garg AK. Free vibration of composite and sandwich laminates with a higher-order facet shell element. Compos Struct 2004; 65: 405-418.

[44] Babu CS, Kant T. Two shear deformable finite element models for buckling analysis of skew fibre-reinforced composite and sandwich panels. Compos Struct 1999; 46: 115-124.

[45] Kant T, Babu CS. Thermal buckling analysis of skew fibre-reinforced composite and sandwich plates using shear deformable finite element models. Compos Struct 2000; 49: 77-85.

[46] Reddy JN. Analysis of functionally graded plates. Int J Numer Methods Eng 2000; 47: 663-684.

[47] Nayak AK, Moy SSJ, Shenoi RA. Free vibration analysis of composite sandwich plates based on Reddy's higher-order theory. Composites: Part B 2002; 33: 505-519.

[48] Nayak AK, Moy SSJ, Shenoi RA. A higher order finite element theory for buckling and vibration analysis of initially stressed composite sandwich plates. J Sound Vib 2005; 286: 763-780.

[49] Sheikh AH, Chakrabarti A. A new plate bending element based on higher-order shear deformation theory for the analysis of composite plates. Finite Element in Analysis and Design 2003; 39: 883-903.

[50] Chakrabarti A, Sheikh AH. Buckling of laminated composite plates by a new element based on higher order shear deformation theory. Mech Advanced Mater Struct 2003; 10: 303-317.

[51] Matsunaga H. Effects of higher-order deformations on in-plane vibration and stability of thick circular rings. Acta Mechanica 1997; 124: 47-61.

[52] Matsunaga H. Vibration and stability of cross-ply laminated composite plates according to a global higher-order plate theory. Compos Struct 2000; 48: 231-244.

[53] Matsunaga H. Vibration and buckling of multilayered composite beam according to higher order deformation theories. J Sound Vib 2001; 246(1): 47-62.

[54] Matsunaga H. Vibration and stability of angle-ply laminated composite plates subjected to in-plane stresses. Int J Mech Sci 2001; 43: 1925-1944.

[55] Matsunaga H. Vibration of cross-ply laminated composite plates subjected to initial in-plane stresses. Thin-Walled Struct 2002; 40: 557-571.

[56] Matsunaga H. Free vibration and stability of laminated composite circular arches subjected to initial axial stress. J Sound Vib 2004; 271: 651-670.

[57] Matsunaga H. Thermal buckling of cross-ply laminated composite and sandwich plates according to a global higher-order deformation theory. Compos Struct 2005; 68: 439-454.

[58] Matsunaga H. Thermal buckling of angle-ply laminated composite and sandwich plates according to a global higher-order deformation theory Compos Struct 2006; 72: 177-192

[59] Matsunaga H. Free vibration and stability of angle-ply laminated composite and sandwich plates under thermal loading. Compos Struct 2007; (77): 249-262.

[60] Matsunaga H. Assessment of a global higher-order deformation theory for laminated composite and sandwich plates. Compos Struct 2002; 56: 279-291.

[61] Matsunaga H. Interlaminar stress analysis of laminated composite beams according to global higher-order deformation theories. Compos Struct 2002; 55: 105-114.

[62] Matsunaga H. Interlaminar stress analysis of laminated composite and sandwich circular arches subjected to thermal/mechanical loading. Compos Struct 2003; 60: 345-358.

[63] Matsunaga H. A comparison between 2-D single-layer and 3-D layerwise theories for computing interlaminar stresses of laminated composite and sandwich plates subjected to thermal loadings. Compos Struct 2004; 64: 161-177.

[64] Cho M, Kim JS. Improved Mindlin plate stress analysis for laminated composite in finite element method. AIAA Stud J 1997; 35: 587-590.

[65] Murakami H. Laminated composite plate theory with improved in-plane response. J Appl Mech 1986; 53: 661-666.

[66] Murakami H, Toledano A. A higher-order laminated plate theory with improved in-plane response. Int J Solids Struct 1987; 23(1): 111-131.

[67] Carrera E. On the use of the Murakami's zig-zag function in the modeling of layered plates and shells. Comput Struct 2004; 82: 541-554.

[68] Carrera E. C^0 Reissner-Mindlin multilayered plate elements including zig-zag and interlaminar stress continuity. Int J Numer Methods Eng 1996; 39: 1797-1820.

[69] Carrera E, Demasi L. Classical and advanced multilayered plate elements based upon PVD and RMVT. Part 1: Derivation of fnite element matrices. Int J Numer Methods Eng 2002; 55: 191-231.

[70] Carrera E, Demasi L. Classical and advanced multilayered plate elements based upon PVD and RMVT. Part 2: Numerical implementations. Int J Numer Methods Eng 2002; 55: 253-291.

[71] Demasi L. Refined multilayered plate elements based on Murakami zig-zag functions. Compos Struct 2005; 70: 308-316.

[72] Carrera E. A priori vs. a posteriori evaluation of transverse stresses in multilayered orthotropic plates. Compos Struct 2000; 48: 245-260.

[73] Carrera E. Transverse normal strain effects on thermal stress analysis of homogeneous and layered plates. AIAA Stud J 2005; 43(10): 2232-2242.

[74] Carrera E, Ciuffreda A. Closed-form solutions to assess multilayered-plate theories for various thermal stress problems. J Thermal Stresses 2004; 27: 1001-1031.

[75] Carrera E, Ciuffreda A. A unified formulation to assess theories of multilayered plates for various bending problems. Compos Struct 2005; 69: 271-293.

[76] Di Sciuva M. Bending, vibration and buckling of simply supported thick multilayered orthotropic plates: An evaluation of a new displacement model. J Sound Vib 1986; 105: 425-442.

[77] Di Sciuva M. Multilayered anisotropic plate models with continuous interlaminar stresses. Compos Struct 1992; 22: 149-167.
[78] Di Sciuva M. A third-order triangular multilayered plate finite element with continuous interlaminar stresses. Int J Numer Methods Eng 1995; 38: 1-26.
[79] Di Sciuva M, Gherlone M. A global/local third-order Hermitian displacement field with damaged interfaces and transverse extensibility: analytical formulation. Compos Struct 2003; 59: 419-431.
[80] Di Sciuva M, Gherlone M. A global/local third-order Hermitian displacement field with damaged interfaces and transverse extensibility: FEM formulation. Compos Struct 2003; 59: 433-444.
[81] Cho M, Parmerter RR. An efficient higher-order plate theory for laminated composites. Compos Struct 1992; 20: 113-123.
[82] Cho M, Oh J. Higher order zig-zag plate theory under thermo-electric-mechanical loads combined. Composites: Part B 2003; 34: 67-82.
[83] Cho M, Oh J. Higher order zig-zag theory for fully coupled thermo- electric-mechanical smart composite plates. Int J Solids Struct 2004; 41: 1331-1356.
[84] Oh J, Cho M. A finite element based on cubic zig-zag plate theory for the prediction of thermo-electric-mechanical behaviors. Int J Solids Struct 2004; 41: 1357-1375.
[85] Cho M, Kim JS, Four-noded finite element post-process method using a displacement field of higher-order laminated composite plate theory. Comput Struct 1996; 61: 283-290.
[86] Cho M, Parmerter RR. Efficient higher order plate theory for general lamination configurations. AIAA J 1993; 31: 1299-1308.
[87] Chakrabarti A, Sheikh AH. A new triangular element to model inter-laminar shear stress continuous plate theory. Int J Numer Methods Eng 2004; 60: 1237-1257.
[88] Chakrabarti A, Sheikh AH. Analysis of laminated sandwich plates based on interlaminar shear stress continuous plate theory. J Eng Mech 2005; 131: 377-384.
[89] Kapuria S. An efficient coupled theory for multilayered beams with embedded piezoelectric sensory and active layers. Int J Solids Struct 2001; 38: 9179-9199.
[90] Kapuria S, Dumir PC, Ahmed A. An efficient coupled layerwise theory for static analysis of piezoelectric sandwich beams. Arch Appl Mech 2003; 73: 147-159.
[91] Kapuria S, Bhattacharyya M, Kumar AN. Assessment of coupled 1D models for hybrid piezoelectric layered functionally graded beams. Compos Struct 2006; 72: 455-468.
[92] Kapuria S, Achary GGS. A coupled zigzag theory for the dynamics of piezoelectric hybrid cross-ply plates. Arch Appl Mech 2005; 75: 42-57.
[93] Kapuria S, Achary GGS. Nonlinear coupled zigzag theory for buckling of hybrid piezoelectric plates. Compos Struct 2006; 74: 253-264.
[94] Kapuria S, Achary GGS. Nonlinear zigzag theory for electrothermo-mechanical buckling of piezoelectric composite and sandwich plates. Acta Mechanica 2006; 184: 61-76.
[95] Kapuria S, Dumir PC, Jain NK. Assessment of zigzag theory for static loading, buckling, free and forced response of composite and sandwich beams. Compos Struct 2004; 64: 317-327.
[96] Kapuria S, Dumir PC, Ahmed A. An efficient higher order zigzag theory for composite and sandwich beams subjected to thermal loading. Int J Solids Struct 2003; 40: 6613-6631.
[97] Kapuria S, Achary GGS. An efficient higher order zigzag theory for laminated plates subjected to thermal loading. Int J Solids Struct 2004; 41: 4661-4684.
[98] Icardi U. Eight-noded zig-zag element for deflection and stress analysis of plates with general lay-up. Composite Part B 1998; 29: 425-441.
[99] Icardi U. C^0 plate element for global/local analysis of multilayered composite, based on a 3D zig-zag model and strain energy updating. Int J Mech Sci 2005; 47: 1561-1594.
[100] Li XY, Liu D. A laminate theory based on global-local superposition. Commun Numer Meth Engng 1995; 11: 633-641.
[101] Sze KY, Chen RG, Cheung YK. Finite element model with continuous transverse shear stress for composite laminates in cylindrical bending. Finite Element in Analysis and Design 1998; 31: 153-164.
[102] Chen RG, Chen WJ. Higher-order shear deformation theory and triangular plate element based on global-local superposition. Acta Materiae Compositae Sinica (in Chinese) 2000; 17: 96-102.
[103] Wu Z, Chen RG, Chen WJ. Refined laminated composite plate element based on global-local higher-order shear deformation theory. Compos Struct 2005; 70: 135-152.
[104] Wu Z, Chen WJ. An efficient higher-order theory and finite element for laminated plates subjected to thermal loading. Compos Struct 2006; 73: 99-109.
[105] Wu Z, Chen WJ. A study of global-local higher-order theory for laminated composite plates. Compos Struct 2007; 79: 44-54.
[106] Rao MK, Desai YM. Analytical solutions for vibrations of laminated and sandwich plates using mixed theory. Compos Struct 2004; 63: 361-373.
[107] Rao MK, Scherbatiuk K, Desai YM, Shah AH. Natural vibrations of laminated and sandwich plates. J Eng Mech 2004; 130: 1268-1278.
[108] Wu Z, Chen WJ. Free vibration of laminated composite and sandwich plates using global-local higher-order theory. J Sound Vib 2006; 298: 333-349.
[109] Wu Z, Chen WJ. Refined global-local higher-order theory and finite element for laminated plates. Int J Numer Methods Eng 2007; 69: 1627-1670.
[110] Pagano NJ. Exact solutions for composite laminates in cylindrical bending. J Compos Mater 1969; 3: 398-411.
[111] Pagano NJ. Exact solutions for rectangular bi-directional composites. J Compos Mater 1970; 4: 20-34.
[112] Dafedar JB, Desai YM. Stability of composite and sandwich struts by mixed formulation. J Eng Mech 2004; 130(7): 762-770.